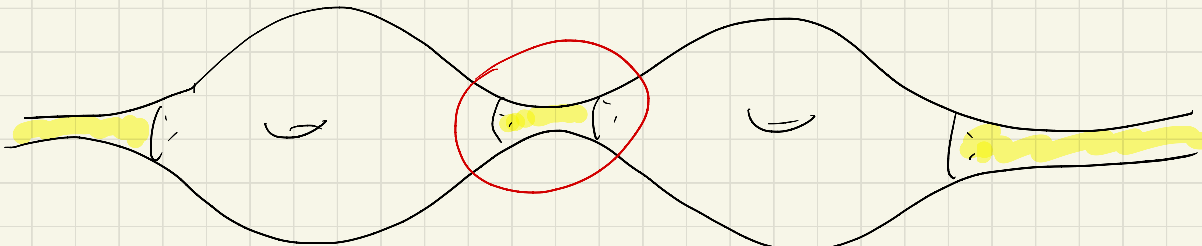
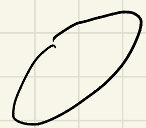
Ogni $M = \mathbb{H}^n / \Gamma$ iperbolica completa si decompone $\quad M = M^{thick} \cup M^{thin}$

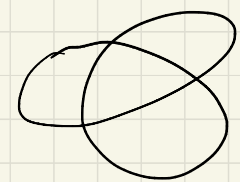


Se $\mathrm{vol}(M) < +\infty \qquad M^{cpt} := M^{thick} \cup \{\text{intorni geod}\}$

$$M = M^{cpt} \cup M^{cusp}.$$

SPETTRO GEODETICO

M hyp $\qquad \Gamma^c \ni [\gamma] \dashrightarrow$  geod. chiusa



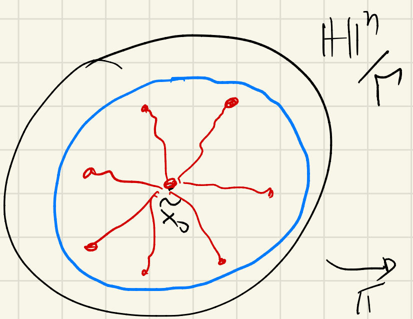
$\underbrace{\{\text{Le lunghezze delle geod. chiuse}\}}_{\text{numerabile}} \subseteq (0, +\infty)$

Prop: $\mathrm{vol}(M) < +\infty \quad \forall L > 0 \quad \exists$ num. finito di geod. chiuse di lunghezza $< L$

dim: M cpt $\to$ diametro finito D

$\exists D : \; d(x,y) < D \quad \forall x, y \in M$

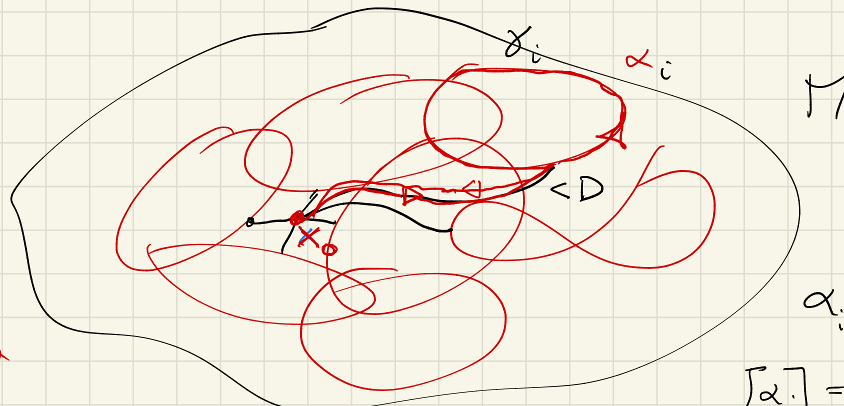
$\gamma_1, \dots, \gamma_i, \dots$ geod. chiuse lunghe $< L$



∞ punti dell'orbita in un cpt

$\not$ ASSURDO

discreta

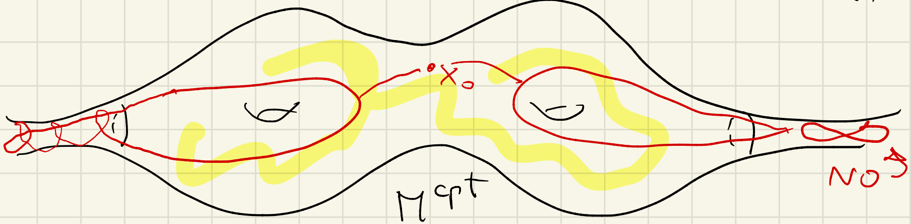


$\alpha_i \in \pi_1(M, x_0)$

$[\alpha_i] = \gamma_i$

diversi $\alpha_1, \dots, \alpha_i, \dots \in \pi_1(M, x_0)$

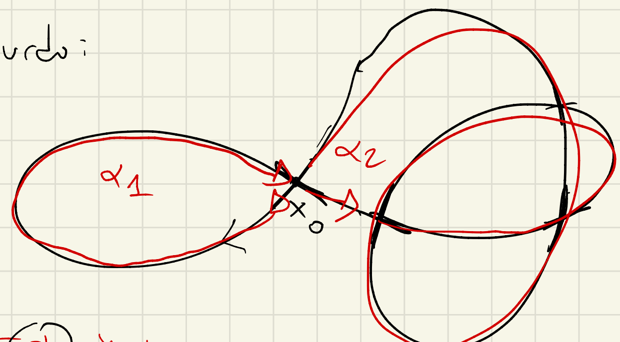
$L(\alpha_i) < 2D + L$

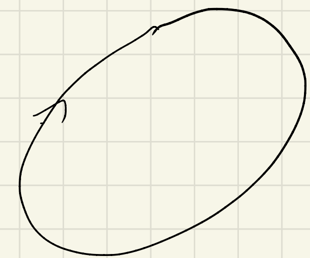


□

Prop: Se M è cpt, every shortest geodesic γ è semplice.

dim: p. assurdo:





$\gamma = [\alpha]$

$\alpha = \alpha_1 * \alpha_2$

PARABOLICI

$\alpha_1 \neq e$ in $\pi_1(M)$

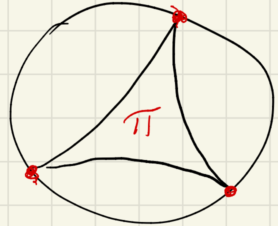
$\alpha_1 \sim \gamma_1$

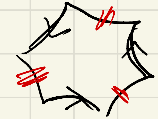
$L(\gamma_1) < L(\alpha_1) < L(\gamma)$

$L(\gamma)$ minimizza nella classe di omotopia

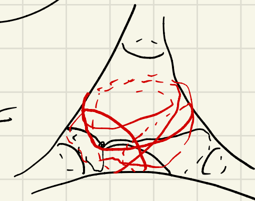
Serve cpt

Esempio:





$\pi_1(S) = \mathbb{Z} * \mathbb{Z}$

$S =$ 

S^2 meno tre punti

sup. ip. area finita

non contiene geodetiche semplici chiuse

ISOMETRIE

Prop: $M = \mathbb{H}^n / \Gamma$

$$\boxed{\mathrm{Isom}(M) \cong N(\Gamma)/\Gamma}$$

$\Gamma < \mathrm{Isom}\, \mathbb{H}^n$

$H < G \qquad N(H)$ normalizzatore

$N(H) = \{ g \in G \mid gH = Hg \}$

cioè $g^{-1} H g = H$

$H \triangleleft N(H) < G$

dim

$$\begin{tikzcd} \mathbb{H}^n \arrow[r, dashed, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \mathbb{H}^n \arrow[d, "\pi"] \\ M \arrow[r, "\varphi"] \arrow[ur, phantom, "\circlearrowright"] & M \end{tikzcd}$$

$\varphi \in \mathrm{Isom}(M) \dashrightarrow \tilde{\varphi} \in \mathrm{Isom}(\mathbb{H}^n)$

$M = \mathbb{H}^n / \Gamma$

$\boxed{\tilde\varphi \Gamma = \Gamma \tilde\varphi}$ $\qquad g \tilde\varphi = \tilde\varphi g'$

$\mathrm{Isom}(M) \longrightarrow N(\Gamma)/\Gamma$ $\qquad \boxed{g \tilde\varphi g' \leftarrow \tilde\varphi}$ $\quad g, g' \in \Gamma$

$\varphi \longmapsto \tilde\varphi$ è iniett. & suriett. $\square$

D: Funziona per $\mathbb{R}^n$ e S^n?

X sp. top. c.p.a. $\to \mathrm{Omeom}(X) \longrightarrow \mathrm{Aut}(\pi_1 X)/\mathrm{Int}(\pi_1 X) = \mathrm{Out}(\pi_1 X)$

$\to \mathrm{Omeom}(X, x_0) \xrightarrow[\text{omom}]{} \mathrm{Aut}(\pi_1(X, x_0))$

$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$

Prop: M ip. vol $< +\infty$ completa $\Rightarrow \mathrm{Isom}(M) \hookrightarrow \mathrm{Omeom}(M) \to \mathrm{Out}(\pi_1 M) = \mathrm{Out}(\Gamma)$

$\mathrm{Isom}(M) = N(\Gamma)/\Gamma \dashrightarrow \mathrm{Out}(\Gamma)$ INIETTIVA

non è vero se M è piatta

dim:

$N(\Gamma)/\Gamma \longrightarrow \boxed{\mathrm{Out}(\Gamma)}$

$g \longmapsto (h \mapsto g^{-1} h g)$

è iniettiva: $f = g \in N(\Gamma) \longmapsto (h \mapsto g^{-1} h g)$ $\in \Gamma$

$g^{-1}hg = f^{-1}hf$ per $f \in \Gamma$

$\forall h \in \Gamma \quad g^{-1} h g = f^{-1} h f \quad f \in \Gamma$

$\Downarrow$

$f g^{-1} h g f^{-1} = h \Rightarrow f g^{-1}$ commuta con h $\forall h \in \Gamma \quad f g^{-1} = h'$

$\Rightarrow f g^{-1} = e \Rightarrow f = g$

Es: $M = \mathbb{H}^n / \Gamma$ vol $< +\infty$

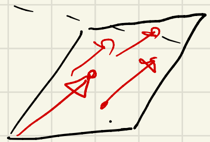
$h' \in \mathrm{Isom}(\mathbb{H}^n)$ che commuta con tutti $h \in \Gamma$

$\Rightarrow h' = e$.

Oss: $T = \mathbb{R}^2 / \Gamma \quad \Gamma \cong \mathbb{Z}^2$

$T < \mathrm{Isom}(T) \longrightarrow \mathrm{Out}(\mathbb{Z}^2)$

Le traslazioni di $\mathbb{R}^2$ si proiettano in T



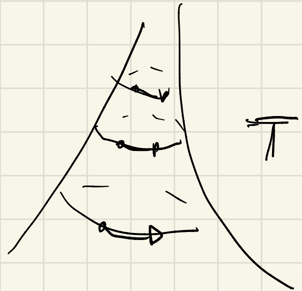
banale

hanno stessa immagine

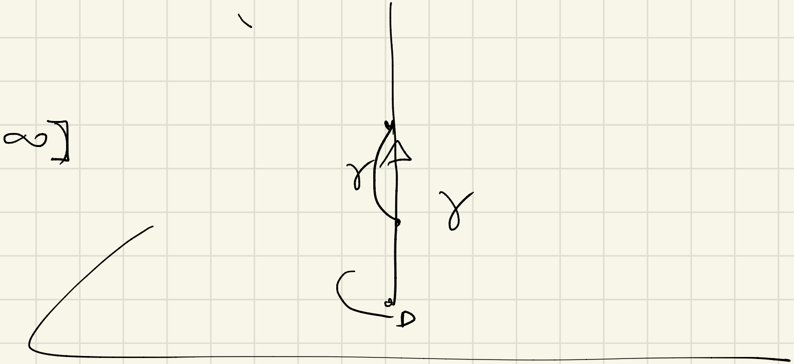
$f \sim g \in \mathrm{Omeo}(X) \longrightarrow \mathrm{Out}(\pi_1 X)$

Cor: $\mathrm{vol}(M) < +\infty \quad f, g \in \mathrm{Isom}(M)$ distinte non sono omotope

Oss: Cuspidi e tubi hanno gruppi di isometria dim > 0



$T \times [0, \infty)$



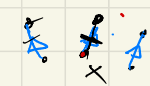
Cor: $M = \mathbb{H}^n / \Gamma$ vol $< +\infty \Rightarrow \mathrm{Isom}(M)$ è finito.

dim: M cpt $\Rightarrow \mathrm{Isom}(M)$ cpt (sempre)

Mostriamo che $\mathrm{Isom}(M)$ è discreto. $\Rightarrow$ finito.

p.a.: $\varphi_i \in \mathrm{Isom}(M)$ $\quad \varphi_i \to id$ $\quad \varepsilon = \mathrm{inj}\, M > 0$

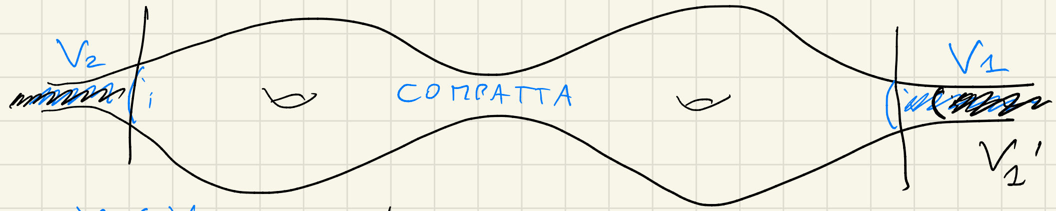
M cpt $\quad \exists\, N : d(\varphi_N(x), x) < \varepsilon \quad \forall x \in M$



$\exists!$ geod. minim. x —— $\varphi_N(x)$

Una geod. minim. $\sim$ $\underline{\underline{id \sim \varphi_N}}$ assurdo. $\square$

M con cuspidi



$V_2 < V_1$ $\qquad V_1' = V_2$

$$\mathrm{Isom}(M) = \mathrm{Isom}(M^{cpt})$$

RESIDUALE FINITEZZA

G residualmente finito $\quad \forall g \neq e$ in G $\quad \exists\, H \triangleleft G$, $g \notin H$
i.f.

$\Gamma < GL(n, \mathbb{C})$ f.g. $\Rightarrow \Gamma$ res. finito

$\exists\, G \longrightarrow F$ finito
$g \longmapsto \neq e$

Cor: $M = \mathbb{H}^n/\Gamma \Rightarrow \Gamma < \mathrm{Isom}(\mathbb{H}^n) \subset GL(N, \mathbb{C})$
è res. finito (se f.g.)

$\mathrm{vol}(M) < +\infty \Rightarrow \Gamma$ res. finito

Cor: M hyp cpt. $\forall L > 0$ $\exists$ $\tilde{M}$ cpt t.c. $\operatorname{inj} \tilde{M} > L$

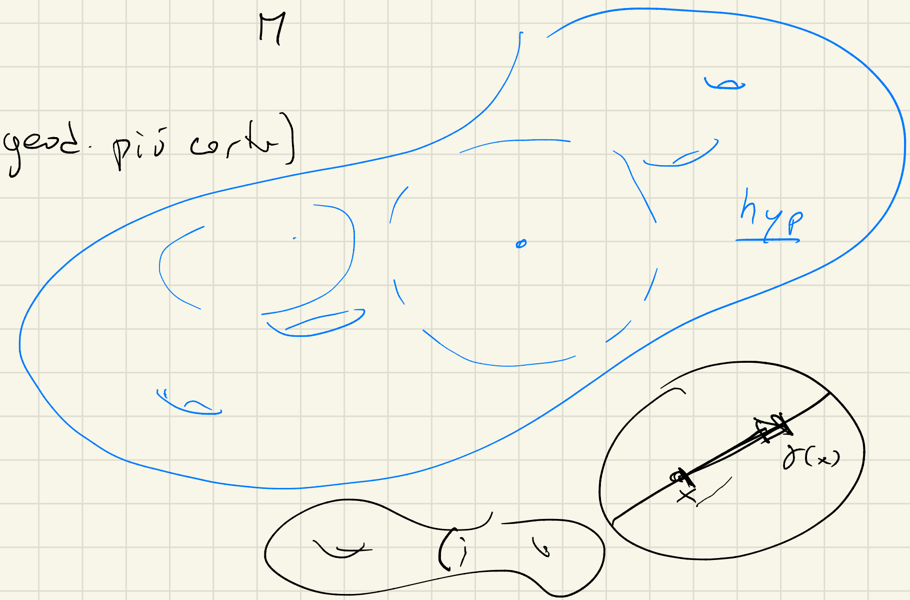
$$\tilde{M} \xrightarrow{\text{finito}} M$$

dim

$$M = \mathbb{H}^n / \Gamma$$

M cpt $\quad \operatorname{inj} M = \frac{1}{2} \{\text{lungh. di geod. più corta}\}$

$$= \frac{1}{2} \left\{ d(\gamma) \;\middle|\; \gamma \in \Gamma,\ \gamma \neq e \right\}$$



geod con lunghezza $< 2L$ sono finite $\gamma_1, \dots, \gamma_r$

$$\gamma_i \sim [\alpha_i] \qquad \alpha_i \in \pi_1 M$$

$$H \triangleleft \Gamma$$

$$\exists\, H \underset{\text{i.f.}}{\triangleleft} \pi_1 M \qquad H \not\ni \alpha_1, \alpha_2, \dots, \alpha_k \neq 0$$

$$H \dashrightarrow \tilde{M}$$

$$\tilde{M} = \mathbb{H}^n / H$$
$$\downarrow$$
$$M = \mathbb{H}^n / \Gamma$$